Behavioral Analyses in Petri Nets by Groebner Bases
— An Application of Ideals and Varieties to Petri Net Problems —

Tadashi MATSUMOTO*, Maki TAKATA* and Seiichiro MORO*

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Finding an non-negative integer solution $x \in Z_+^{n\times 1}$ for $Ax = b\,(A \in Z^{m\times n},\, b \in Z^{m\times 1})$ in Petri nets is NP-complete. Being NP-complete, even algorithms with theoretically bad worst case and with average complexity can be useful for a special class of problems, hence deserve investigation. Then a Groebner basis approach to integer programming problems was proposed in 1991 and some symbolic computation systems became to have useful tools for ideals, varieties, and algorithms for algebraic geometry. In this paper, two kinds of examples are given to show how Groebner basis approach is applied to reachability problems in Petri nets.

Key words: Integer Programming, Groebner Basis, Buchberger Algorithm, Algebraic Geometry, Petri Net Reachability Problems, Symbolic Computation Systems

1. Introduction

It has been well recognized that Petri nets are one of useful models to represent and analyze discrete event systems. However, finding a non-negative integer solution (i.e., a firing count vector) for state equation of Petri nets is to solve an integer programming problem.[1]

On the other hand, the problem of finding a solution (all solutions) in non-negative (positive) integers for a system of linear equations with integer coefficients is a classical problem which is NP-complete. If moreover we have a linear cost function (a linear map with real coefficients) and we look for an optimal solution, the problem is an equivalent form of the integer programming problem. Being NP-complete, even algorithms with theoretically bad worst case and with average complexity can be useful for a special class of problems, hence deserve investigation. The algorithm, which we consider, that transforms a linear system problem into a Groebner basis problem for a binomial ideals falls in this class of algorithms[2], [3], [4], [5]. Note that an algorithm (i.e., Gauss's elimination algorithm) for finding a solution for a system of linear equations over polynomial ring with respect to one-variable is extended to that over polynomial ring with respect to multi-variables by choosing Groebner bases as ideal bases.

Although any direct and systematic method to find an exact firing count vector for state equation of Petri nets has not been known, this Groebner basis approach can give a useful direct and systematic method to Petri net reachability problems. In this paper, two kinds of examples are given to show how Groebner basis approach is applied to state equation of Petri nets.

(1) Find all particular solutions as well as all generators for T-invariants.
(2) Find all possible representations of an arbitrary firing count vector by using all generators for T-invariants and a particular solution.

2. Preliminaries [4], [5]

Consider a polynomial in n variables $y_1, \cdots, y_n$, where each coefficient belongs to any field k.

Definition 2.1

Let us call $y_1^{\alpha_1} \cdot y_2^{\alpha_2} \cdots y_n^{\alpha_n}$ a monomial of $y_1, \cdots, y_n$, where $\alpha_1, \cdots, \alpha_n$ are nonnegative integers. The sum $\alpha_1 + \cdots + \alpha_n$ is said to be a total degree for the above monomial. ■

* Dept. of Electrical and Electronics Engineering

Definition 2.2

A linear combination of a finite number of monomials with coefficients on any field k is called a polynomial in $y_1,\cdots,y_n$, where each coefficient belongs to field k. That is, a polynomial f is expressed by

$$f=\sum_\alpha a_\alpha y^\alpha, \quad a_\alpha\in k,$$

where summations is done for a finite number of $\alpha=(\alpha_1, \cdots,\alpha_n)^T$. Let $k[y_1,\cdots,y_n]$ be the set of all polynomials in $y_1,\cdots,y_n$ where each coefficient is over k. ■

Definition 2.3

The subset $I\subset k[y_1,\cdots,y_n]$ is an ideal if and only if the following properties are satisfied:

(i) $0\in I$.

(ii) If $f, g\in I$, then $f+g\in I$.

(iii) If $f\in I$ and $h\in k[y_1,\cdots,y_n]$, then $hf\in I$.

■

Definition 2.4

Let $f_1,\cdots,f_s$ be polynomials such that $f_1,\cdots,f_s \in k[y_1,\cdots,y_n]$. Then we define

$$<f_1,\cdots,f_s>=\left\{\sum_{i=1}^s h_i f_i : h_1,\cdots,h_s\in k[y_1,\cdots,y_n]\right\}.$$

■

It is important that $<f_1,\cdots,f_s>$ is an ideal.

Definition 2.5

We call $>$ a monomial order on $k[y_1,\cdots,y_n]$ if $>$ have the following properties:

(i) $>$ is a total order over Z_+^n.

(ii) $\alpha>\beta \Rightarrow \alpha+\gamma>\beta+\gamma$ for all $\gamma\in Z_+^n$ (i.e., $>$ is compatible with sums).

(iii) $\alpha>0$ for all $\alpha\in Z_+^n\setminus\{0\}$ (i.e., 0 is the minimum element.). ■

Definition 2.6 (Lexicographic Order $>_{lex}$)

For $\alpha=(\alpha_1,\cdots,\alpha_n)^T\in Z_+^n$ and $\beta=(\beta_1,\cdots,\beta_n)^T \in Z_+^n$, $\alpha>_{lex}\beta$ means that the non-zero element on the most left of $\alpha-\beta\in Z_+^n$ is positive. When $\alpha>_{lex}\beta$ is satisfied, we denote it $y^\alpha>_{lex}y^\beta$. ■

Definition 2.7

Let $f=\sum_\alpha a_\alpha y^\alpha\in k[y_1,\cdots,y_n]$ be a nonzero polynomial and $>$ be a monomial order on $k[y_1,\cdots,y_n]$. Then (i)~(iv) are defined as follows.

(i) A multidegree for f is given by

$$\text{multideg}(f)=\max(\alpha\in Z_+^n : a_\alpha\neq 0),$$

in which maximization is done with respect to order $>$.

(ii) A leading coefficient is given by

$$LC(f)=\alpha_{\text{multideg}(f)}\in k.$$

(iii) A leading monomial of f is

$$LM(f)=y^{\text{multideg}(f)}.$$

(iv) A leading term of f is

$$LT(f)=LC(f)\cdot LM(f).$$ ■

[Theorem 1]

(Algorithm for Division in $k[y_1,\cdots,y_n]$)

Fixed a monomial order $>$ over Z_+^n, let $F=(f_1,\cdots, f_s)$ be a set of s polynomials ordered on $k[y_1,\cdots, y_n]$. Then any $f\in k[y_1,\cdots,y_n]$ is written as $f=a_1f_1+\cdots+a_sf_s+r$, where $a_i\ r\in k[y_1,\cdots, y_n]$. Note that r is 0 or a linear combination of monomials with coefficients over k, in which each monomial is not completely divided by any of $LT(f_1),\cdots,LT(f_s)$. Let r be called a remainder when f is divided by F. Let us denote it as $r=\overline{f}^F$ in this paper. If $a_if_i\neq 0$ is true, then

$$\text{multideg}(f)\geqq \text{multideg}(a_if_i).$$ ■

3. Groebner Bases [4], [5]

Definition 3.1

Fixed a monomial order $>$ on a polynomial ring $k[y_1,\cdots,y_n]$, let $I\subset k[y_1,\cdots,y_n]$ be an ideal. Then a Groebner basis of I with respect to $>$ is a finite set of polynomials $G=\{g_1,\cdots,g_t\}\subset I$ such that, for each and non-zero $f\in I$, $LT(f)$ is completely divided by some $LT(g_i)$. ■

Definition 3.2

Assume that polynomials $f,\ g\in k[y_1, \cdots,y_n]$ are non-zero. Fixed a monomial order, we assume that

$$LT(f)=cy^\alpha,\ LT(g)=dy^\beta,\ c,d\in k,$$

are satisfied. Let y^γ be the least common multiple for y^α and y^β. Then

$$S(f,g)=\frac{y^\gamma}{LT(f)}\cdot f-\frac{y^\gamma}{LT(g)}\cdot g$$

is said to be an S-polynomial for f and g. ■

Note that $S(f,g)\in<f,g>$ due to Definition 3.2. An important result for a process which constructs a remainder for an S-polynomial is known as follows.

<Buchberger Criterion>

A finite set $G=\{g_1,\cdots,g_t\}$ is a Groebner basis of $I=<g_1,\cdots,g_t>$ if and only if $\overline{S(g_i,g_j)}^G=0$ for all pairs of g_i and g_j s.t. $i\neq j$. ■

[Theorem 2]

(Groebner Basis and Buchberger Algorithm)

Let $I = < f_1, \cdots, f_s > \neq \{0\}$ be a polynomial ideal. A Groebner basis of I is constructed in a finite steps by the following Buchberger algorithm.

Input: $F = (f_1, \cdots, f_s)$

Output: a Groebner basis $G = (g_1, \cdots, g_t)$ for I, with $F \subset G$

$G := F$

REPEAT

$G' := G$

FOR each pair $\{p, q\}$, $p \neq q$ in G' DO

$S := \overline{S(p,q)}^{G'} = 0$

IF $S \neq 0$ THEN $G := G \cup \{S\}$

UNTIL $G = G'$ ■

4. Solving IP Problems by Groebner Bases [4]

We want to solve the system

$$Ax = b, \ A \in Z^{m \times n}, \ b \in Z^{m \times 1}, \qquad (1)$$

finding a solution $x \in Z_+^n$ such that the value of a linear cost function $c^T x$ is minimal, $c^T \in R^n$.

This is called as a standard form for an integer programming problem.

4.1 Special IP Problems with $a_{ij} \geqq 0$ and $b_i \geqq 0$

The problem is translated twice, obtaining a problem that is solvable by Buchberger algorithm. To explain this translation, we drop the minimality assumption and consider first the special case $a_{ij} \geqq 0$ and $b_i \geqq 0$, where $A = [a_{ij}]$ and $b = [b_i]$.

Introducing an indeterminant z_i to each scalar equation of Eq.(1), we have

$$\prod_{j=1}^{n} (\prod_{i=1}^{m} z_i^{a_{ij}})^{x_j} = \prod_{i=1}^{m} z_i^{b_i} \qquad (2)$$

From Eq.(2), we obtain a direct and algebraic characteristic about one pair of n integer variables in the feasible region defined by Eq.(1), as follows.

[Theorem 3]

Fixed a field k and for $j = 1, \cdots, n$ let us first define

$$\varphi(w_j) = \prod_{i=1}^{m} z_i^{a_{ij}}.$$

Secondly, define

$$\varphi : \ k[w_1, \cdots, w_n] \ \rightarrow \ k[z_1, \cdots, z_m],$$

such that $\varphi(g(w_1, \cdots, w_n)) = g(\varphi(w_1), \cdots, \varphi(w_n))$ is assumed for a general polynomial $g \in k[w_1, \cdots, w_n]$. Then, there exists $x = (x_1, \cdots, x_n)^T$ on an integer point in the feasible region defined by Eq.(1) if and only if φ maps from a monomial $w_1^{x_1} w_2^{x_2} \cdots w_n^{x_n}$ to a monomial $z_1^{b_1} z_2^{b_2} \cdots z_m^{b_m}$. ■

The following test for an element of image of the mapping φ in Theorem 3 is one of important facts about translation for IP: Since an image of φ is a set of polynomials in $k[z_1, \cdots, z_m]$ which is expressed by polynomials $f_j = \prod_{i=1}^{m} z_i^{a_{ij}}$, we can write the image of φ as the subring $k[f_1, \cdots, f_n]$ of $k[z_1, \cdots, z_m]$ which is generated by f_j.

[Theorem 4]

Consider polynomials $f_1, \cdots, f_n \in k[z_1, \cdots, z_m]$. First, assume that a monomial order for $k[z_1, \cdots, z_m, w_1, \cdots, w_n]$ is determined such that every z_i is larger than any power of the w_j. This is called an elimination monomial order. Secondly, let g be a Groebner basis of an ideal

$$I = < f_1 - w_1, \cdots, f_n - w_n > \subset k[z_1, \cdots, z_m, w_1, \cdots, w_n]$$

and let $\overline{f}^{\mathrm{g}}$ be a remainder of f divided by g for each $f \in k[z_1, \cdots, z_m]$.

Then, we have the next three statements.

a. A polynomial f fulfills $f \in k[f_1, \cdots, f_n]$ if and only if $g = \overline{f}^{\mathrm{g}} \in k[w_1, \cdots, w_n]$.

b. Let $f \in k[f_1, \cdots, f_n]$ and $g = \overline{f}^{\mathrm{g}} \in k[f_1, \cdots, f_n]$ be those defined in the above a. Then, f which is expressed by a polynomial in f_j satisfies $f = g(f_1, \cdots, f_n)$.

c. When each f_j and f are monomials $f \in k[f_1, \cdots, f_n]$, $g = \overline{f}^{\mathrm{g}}$ is also a monomial. ■

Note that the statement c in Theorem 4 guarantees that when $z_1^{b_1} \cdots z_m^{b_m}$ is an image of φ under the conditions in Theorem 3, it is always an image of a monomial $w_1^{x_1} \cdots w_n^{x_n}$.

Next, in order to find an optimal solution for IP which minimizes the given linear cost function $c^T x$, we should in general adopt a specifically-adjusted monomial order which is adapted to the given specific problem. Then we have the next definition.

Definition 4.1

If a monomial order on $k[z_1, \cdots, z_m, w_1, \cdots, w_n]$ has the following two properties, then it is said to be adapted to the integer programming problem defined by Eq.(1):

a. (Elimination Property) Any monomial containing at least one of z_i is larger than every monomial contain- ing only w_j.
b. (Compatibility with $c^T x$) Let $x=(x_1,\ldots,x_n)^T$ and $x'=(x'_1,\ldots,x'_n)^T$. If monomials w^x and $w^{x'}$ satisfy $\varphi(w^x)=\varphi(w^{x'})$ and $c^T x > c^T x'$, then $w^x > w^{x'}$. ■

Then we have the next property.

[Theorem 5]

Consider IP problem defined by Eq.(1). Assume that $a_{ij}, b_i \geqq 0$ for all i, j and let $f_j = \prod_{i=1}^m z_i^{a_{ij}}$ mean one defined previously. Consider also a Groebner basis g of the ideal $I = < f_1 - w_1, \cdots, f_n - w_n > \subset k[z_1, \cdots, z_m, w_1, \cdots, w_n]$ which is defined with respect to an adapted monomial order. Now, if $f = z_1^{b_1} \cdots z_m^{b_m}$ is an element of $k[f_1, \cdots, f_n]$, then $\overline{f}^{\mathrm{g}} \in k[w_1, \cdots, w_n]$ gives a solution for IP problem which minimizes $c^T x$. ■

From Theorem 5, we have an algorithm for solving IP problems with $a_{ij}, b_j \geqq 0$ for all i, j by using Groebner bases.

<Algorithm for IP Problems>

Input: A, b of Eq.(1), an adapted monomial order $>$.

Output: a solution of IP problem, if one exists

$f_j := \prod_{i=1}^m z_i^{a_{ij}}$

$I := < f_1 - w_1, \cdots, f_n - w_n >$

g := Groebner basis of I with respect to $>$

$f := \prod_{i=1}^m z_i^{b_i}$

$g := \overline{f}^{\mathrm{g}}$

IF $g \in k[w_1, \cdots, w_n]$ THEN

its exponent vector gives a solution

ELSE

there is no solution ■

4.2 General IP Problems with $a_{ij} \gtreqqless 0$ and $b_i \gtreqqless 0$

Consider the general IP problems with $a_{ij} \gtreqqless 0$ and $b_i \gtreqqless 0$. Note that there exists no conceptual difference between the special IP problem given in section 4.1 and the general one in this subsection. Geometric interpretation is the same for both IP problems, i.e., a position which determines the feasible region over affine vector space is just changed in the general case. However, algebraic interpretation is not the same for both IP problems, i.e., negative a_{ij} and b_i are not exponents in general IP problems. This issue is overcome by considering a Laurent polynomial in z_i, i.e., a polynomial in z_i and z_i^{-1}. An expression for a Laurent polynomial ring is used such as

$k[z_1^{\pm 1}, \cdots, z_m^{\pm 1}] \simeq k[z_1, \cdots, z_m, t] / < t z_1 z_2 \cdots z_m - 1 >$,

where a Laurent polynomial ring is represented as a quotient ring of polynomial rings. Note that this isomorphism($\simeq$) goes well by introducing just a variable t which satisfies $t z_1 z_2 \cdots z_m - 1 = 0$.

$\prod_{i=1}^m z_i^{a_{ij}}$ and $\prod_{i=1}^m z_i^{b_i}$ in Eq.(2) is rewritten as $t^{e_j} \prod_{i=1}^m z_i^{a'_{ij}}$ ($e_j \geqq 0$, $a'_{ij} \geqq 0$ for all i, j, and $a'_{ij} = a_{ij} + e_j$) and $t^e \prod_{i=1}^m z_i^{b'_i}$ ($e \geqq 0$, $b'_i \geqq 0$ for all i, and $b'_i = b_i + e$), respectively. Therefore Eq.(2) is changed to the expression which is obtained by modifying

$$\prod_{j=1}^n (t^{e_j} \prod_{i=1}^m z_i^{a'_{ij}})^{x_j} = t^e \prod_{i=1}^m z_i^{b'_i}$$

using module for $t z_1 z_2 \cdots z_m - 1 = 0$.

Then Theorem 3 is extended as follows.

[Theorem 6]

For each $j = 1, \cdots, n$, let us define

$\varphi(w_i) = t^{e_j} \prod_{i=1}^m z_i^{a'_{ij}} \bmod < t z_1 \cdots z_m - 1 >$.

Define also the next mapping

$\varphi : k[w_1, \cdots, w_n] \to k[z_1^{\pm 1}, \cdots, z_m^{\pm 1}]$

by extending $\varphi(w_i)$ to a general polynomial $g(w_1, \cdots, w_n) \in k[w_1, \cdots, w_n]$ as in section 4.1.

Then $x = (x_1, \cdots, x_n)^T$ is an integer point on the feasible region defined by Eq.(1) if and only if $\varphi(w_1^{x_1} w_2^{x_2} \cdots w_n^{x_n})$ and $t^e z_1^{b_1} \cdots z_m^{b_m}$ mean one same element in $k[z_1^{\pm 1}, \cdots, z_m^{\pm 1}]$, in other words, the difference between them is completely divided by $t z_1 \cdots z_m - 1$. ■

Theorem 4 is also extended as follows.

[Theorem 7]

Consider polynomials $f_1, \cdots, f_n \in k[z_1, \cdots, z_m, t]$.

First, assume that a monomial order for $k[z_1, \cdots, z_m, t, w_1, \cdots, w_n]$ is determined such that a monomial containing at least one of $z_1, \cdots, z_m, t$ is larger than every monomial containing only $w_1, \cdots, w_n$. This is also called an elimination monomial order. Secondly, let g be a Groebner basis of an ideal

$$J = < t z_1 z_2 \cdots z_m - 1, f_1 - w_1, \cdots, f_n - w_n >$$

and let $\overline{f}^{\mathrm{g}}$ be a remainder of f divided by g for each $f \in k[z_1, \cdots, z_m, t]$.

Then, we have the next three statements.

a. A polynomial f is an element of S (i.e., the set of S-polynomials, see Definition 3.2) if and only if $g = \overline{f}^{\mathrm{g}} \in k[w_1,\cdots,w_n]$.

b. As in the above statement a, we assume that f is an element of S and $g = \overline{f}^{\mathrm{g}} \in k[w_1,\cdots,w_n]$. Then f which is expressed by a polynomial in f_j satisfies $f = g(f_1,\cdots,f_n)$.

c. When each f_j and f are monomials and $f \in S$, g is also a monomial. ■

In order to find an optimal solution for general IP problems, we should extend Definition 4.1 and Theorem 5 in section 4.1. The details for those are given in [2], [3].

5. Applications of Groebner Basis Approach to Petri Nets

5.1 General Remarks about Groebner Basis Approach

(1) For simplicity, we drop the minimality assumption for IP in this paper when we apply Groebner basis approach to Petri net problems. Then we use < Algorithm for IP Problems> of section 4.1 in which, however, an adapted monomial order is replaced with a regular one and I, f_j, and f are also replaced with J, f_j, and f of Theorem 7, respectively, because of $a_{ij} \gtreqless 0$ and $b_i \gtreqless 0$ in Petri nets.

(2) We use the symbolic computation system Maple 7[4], [5], [6] in order to calculate g, Groebner Bases, for J or I and to divide $f \in k[z_1,\cdots,z_m,t]$ by g in <Algorithm for IP problems>. See section 5.2.1.

(3) For T-invariants of Petri nets, we must solve $Ax = 0^{m\times 1}$. Then we have $f = 1$ in this case. Since we can not divide $f = 1$ by g as in (2), we will pay attention to a special Groebner basis in order to have T-invariants. See section 5.2.2.

(4) One example in Petri nets for the special IP problems of section 4.1 will be given in section 5.3.

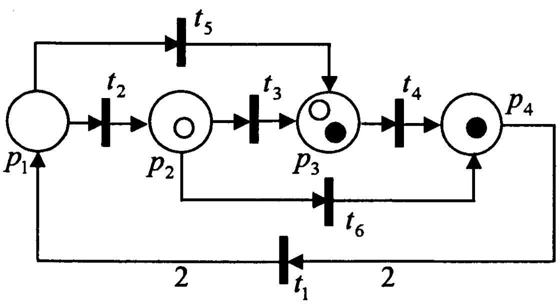

Fig.1 A simple Petri net

5.2 Some Applications of General IP Problems with $a_{ij} \gtreqless 0$ and $b_i \gtreqless 0$ to Petri Nets Problems

Consider a Petri net shown in Fig.1, where ● (○) in a place is the initial (final) marking. Then we have

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ -2 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \in Z^{4\times 6} \text{ and } b = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \in Z^{4\times 1}.$$

Therefore for this example we use the general IP problems shown in section 4.2.

5.2.1 Finding All Particular Solutions

① Define the mapping $\varphi : k[w_1, w_2, \cdots, w_6] \to k[z_1^{\pm 1},\cdots,z_4^{\pm 1}]$ for $Ax = b$. Then from $Ax = b$ and $x = (x_1,\cdots,x_6)^T$ we have

$$z_1^{2x_1-x_2-x_5} = z_1^0,\quad z_2^{x_2-x_3-x_6} = z_2,$$
$$z_3^{x_3-x_4+x_5} = z_3^0,\quad z_4^{-2x_1+x_4+x_6} = z_4^{-1},$$

and have also the followings:

$$\varphi(w_1) = \textstyle\prod_{i=1}^4 z_i^{a_{i1}} = z_1^2 z_4^{-2},\quad \varphi(w_2) = \prod_{i=1}^4 z_i^{a_{i2}} = z_1^{-1} z_2,$$
$$\varphi(w_3) = \textstyle\prod_{i=1}^4 z_i^{a_{i3}} = z_2^{-1} z_3,\ \varphi(w_4) = \prod_{i=1}^4 z_i^{a_{i4}} = z_3^{-1} z_4,$$
$$\varphi(w_5) = \textstyle\prod_{i=1}^4 z_i^{a_{i5}} = z_1^{-1} z_3,\ \varphi(w_6) = \prod_{i=1}^4 z_i^{a_{i6}} = z_2^{-1} z_4.$$

② Applying $t = z_1^{-1}z_2^{-1}z_3^{-1}z_4^{-1}$ (i.e., $tz_1z_2z_3z_4 - 1 = 0$) to $\varphi(w_j)$ in ①, we obtain

$$w_1 \mapsto f_1 = t^2 z_1^4 z_2^2 z_3^2,\quad w_2 \mapsto f_2 = tz_2^2 z_3 z_4,$$
$$w_3 \mapsto f_3 = tz_1 z_3^2 z_4,\quad w_4 \mapsto f_4 = tz_1 z_2 z_4^2,$$
$$w_5 \mapsto f_5 = tz_2 z_3^2 z_4,\quad w_6 \mapsto f_6 = tz_1 z_3 z_4^2.$$

③ Then we have an ideal for ② as

$$J = \langle tz_1z_2z_3z_4 - 1,\ f_1 - w_1,\ \cdots,\ f_6 - w_6 \rangle.$$

④ Groebner Bases for J are obtained, in 5.5 second, by using the symbolic computation system Maple 7 as $\mathrm{g} = \{g_1,\cdots,g_{10}\}$, where

$$g_1 = w_3w_4 - w_6,$$
$$g_2 = w_2w_6 - w_4w_5,\quad g_3 = w_2w_3 - w_5,$$
$$g_4 = w_1w_5^2w_6^2 - w_3^2,\quad g_5 = w_1w_4w_5^2w_6 - w_3,$$
$$g_6 = w_1w_4^2w_5^2 - 1,\quad g_7 = z_3 - w_1w_4w_5^2z_4,$$
$$g_8 = z_2 - w_1w_2w_4w_5z_4,\ g_9 = z_1 - w_1w_4w_5z_4,$$
$$g_{10} = tz_4^4 - w_4^2w_5w_6,$$

and the lexicographic order defined in Definition 2.6 is used, i.e., $t >_{lex} z_1 >_{lex} \cdots >_{lex} z_4 >_{lex} w_1 >_{lex} \cdots >_{lex} w_6$.

⑤ Since $b \neq 0$, we have $f = \prod_{i=1}^4 z_i^{b_i} = z_2 z_4^{-1}$. If we apply $t = z_1^{-1}z_2^{-1}z_3^{-1}z_4^{-1}$ to f, then we have

$f = 1, \in k[z_1, \cdots, z_4, t]$.

⑥ If we apply Maple 7 to $f = tz_1 z_2^2 z_3$, in order to obtain $\overline{f}^{\mathrm{g}}$, then we have $g^{(1)} = \overline{f}^{\mathrm{g}} = w_1 w_2 w_4 w_5 \in k[w_1, w_2, \cdots, w_6]$ in 0.1 second
and $v_1 := x = (1\,1\,0\,1\,1\,0)^T$, where the lexicographic order is also used. These are due to Theorems 6 and 7.

⑦ If we apply $g_2 = w_2 w_6 - w_4 w_5 = 0$ to $g^{(1)} = w_1 w_2 w_4 w_5$, then we have $g^{(2)} = w_1 w_2^2 w_6$ and $v_2 := x = (120001)^T$. We also have $g^{(3)} = w_1 w_2^2 w_3 w_4$ and $v_3 := x = (121100)^T$ by applying $g_3 = w_2 w_3 - w_5 = 0$ to $g^{(1)} = w_1 w_2 w_4 w_5$.

Therefore we have all particular solutions for this example at level 5; $\{v_1, v_2, v_3\} = V_5$[7].

Note that V_5 is also the set of minimal support vectors for this special example.

5.2.2 Finding All Minimal T-invariants

① We want to have generators for $x \in Z_+^{6\times 1}$ in $Ax = 0^{6\times 1}$. Then we have the same set of Groebner bases for this example as that in ④ in section 5.2.1.

② Since $Ax = b = 0^{6\times 1}$, we have $f = \prod_{i=1}^4 z_i^{b_i} = 1$. However, as has been pointed out in (3) of section 5.1, we have the special Groebner basis $g_6 = w_1 w_4^2 w_5^2 - 1$ in this case. Then if we apply $g_6 = 0$ to $f = 1$, then we have $f^{(1)} = w_1 w_4^2 w_5^2 = f$ and also have
$u_1 := x = (100220)^T$ which are due to Theorem 6.

③ If we apply $g_2 = 0$ to $f = w_1 w_4^2 w_5^2$, we have the next two cases;
$f^{(2)} = w_1 w_2^2 w_6^2 = f$ and $u_2 := x = (120002)^T$,
$f^{(3)} = w_1 w_2 w_4 w_5 w_6 = f$ and $u_3 := x = (110111)^T$.

④ We also have the next two cases;
$f^{(4)} = w_1 w_2^2 w_3^2 w_4^2 = f$ and $u_4 := x = (122200)^T$,
$f^{(5)} = w_1 w_2 w_3 w_4^2 w_5 = f$ and $u_5 := x = (111210)^T$,
by applying $g_3 = 0$ to $f = w_1 w_4^2 w_5^2$.

⑤ Moreover if we apply $g_1 = 0$ to $f^{(2)} = w_1 w_2^2 w_6^2$ in ③, we obtain $f^{(6)} = w_1 w_2^2 w_3 w_4 w_6 = f$ and $u_6 := x = (121101)^T$.

⑥ Therefore through Groebner basis approach we had the set of all minimal support T-invariants $U_4 = \{u_1, u_2, u_4\}$ and $U_5 \setminus U_4 = \{u_3, u_5, u_6\}$, where U_5 means the set of all minimal T-invariants and $u_3 = (u_1 + u_2)/2$, $u_5 = (u_1 + u_4)/2$, $u_6 = (u_2 + u_4)/2$ in this special example. Note that the facts about U_4 and U_5 were given by another indirect methods[7].

⑦ If we apply repeatedly $g_4 = 0$ and $g_5 = 0$ to $f^{(4)}, f^{(5)}, f^{(6)}$, we have many T-invariants which are not minimal.

⑧ It is needed for T-invariants to guarantee the existence of a special Groebner basis such as g_6.

5.3 An Application of Special IP Problems with $a_{ij} \geqq 0$ and $b_i \geqq 0$ to Petri Net Problems

Problem Description[8]: Suppose that $U_5, V_5 \subset Z_+^{n\times 1}$ are given as in section 5.2. Find all level 5 representations for the specified firing count vector $x \in Z_+^{n\times 1}$ with respect to U_5 and V_5 such that
$x = \sum_{i=1}^{l_5} \alpha_i u_i + v_j$, $u_i \in U_5$, $v_j \in V_5$, $\alpha_i \in Z_+^{1\times 1}$, $i \in \{1, \cdots, l_5\}$, $j \in \{1, \cdots, k_5\}$; in other words, find all possible $\alpha := (\alpha_1, \cdots, \alpha_{l_5})^T \in Z_+^{l_5 \times 1}$.
This is written as $A\alpha = b$, where
$A := [u_1, \cdots, u_{l_5}] \in Z_+^{n\times l_5}$, $b := x - v_j \in Z_+^{n\times 1}$ ■

Therefore this problem belongs to the special IP problems and $\alpha \in Z_+^{l_5\times 1}$ is found by <Algorithm for IP Problems> in section 4.1, but replacing an adapted monomial order with a regular one.

① Consider again the example of Fig.1. We have found the set of minimal T-invariants $U_5 = \{u_1, \cdots, u_6\}$ and the set of particular solutions $V_5 = \{v_1, v_2, v_3\}$ for $Ax = b$ in section 5.2, wherein $u_i \in U_5$ and $v_j \in V_5$ are also given. Then we have $A\alpha = b$, where $A \in Z_+^{6\times 6}$, $b \in Z_+^{6\times 1}$, and we want to have $\alpha \in Z_+^{6\times 1}$.

② First, define $\varphi : k[w_1, \cdots, w_6] \to k[z_1, \cdots, z_6]$ for $A\alpha = b$. Then we have $w_1 \mapsto f_1 = z_1 z_4^2 z_5^2$, $w_2 \mapsto f_2 = z_1 z_2^2 z_6^2$, $w_3 \to f_3 = z_1 z_2 z_4 z_5 z_6$, $w_4 \mapsto f_4 = z_1 z_2^2 z_3^2 z_4^2$, $w_5 \mapsto f_5 = z_1 z_2 z_3 z_4^2 z_5$, $w_6 \mapsto z_1 z_2^2 z_3 z_4 z_6$.
An ideal is given by
$I = \langle f_1 - w_1, f_2 - w_2, \cdots, f_6 - w_6 \rangle$.

③ A set of Groebner bases for I is calculated by Maple 7 as $\mathrm{g} = \{g_1, \cdots, g_{24}, g_{25}, \cdots, g_{30}\}$, where $g_{25} = z_1 z_4^2 z_5^2 - w_1$, $g_{26} = z_1 z_2 z_4 z_5 z_6 - w_3$, $g_{27} = z_1 z_2 z_3 z_5^2 - w_5$, $g_{28} = z_1 z_2^2 z_6^2 - w_2$, $g_{29} = z_1 z_2^2 z_3 z_4 z_6 - w_6$, $g_{30} =$

$z_1 z_2^2\ z_3^2 z_4^2 - w_4$.

④ Assume $x = (4\ 4\ 1\ 5\ 4\ 2)^T$ and v_2 = $(1\ 1\ 0\ 1\ 1\ 0)^T$ are set to Fig.1, then we have b = $(3\ 3\ 1\ 4\ 3\ 2)^T$.

For $b \in Z_+^{6\times1}$, we have $f = z_1^3 z_2^3 z_3 z_4^4 z_5^3 z_6^2$.

⑤ When Maple 7 is applied to f, we have $g^{(1)} = \overline{f}^{\mathrm{g}} = w_3^2 w_5$. Since $w_3^2 w_5 \in k[w_1,\cdots,w_6]$, then we have an solution $\alpha^{(1)} = (0\ 0\ 2\ 0\ 1\ 0)^T$ from $w_3^2 w_5$. This means that x is represented as $x = 2u_3 + u_5 + v_2$.

⑥ Note that $w_1, w_2, \cdots, w_6$ obtained from $g_i = 0$ $(i = 25,\cdots,30)$ in ③ are corresponding to $u_1,\cdots,u_6$. Then if we apply $w_2,\cdots,w_6$ to f in ④, we have other solutions as $g^{(2)} = w_1 w_2 w_5$ (*i.e.*, $\alpha^{(2)} = (1\ 1\ 0\ 0\ 1\ 0)^T$ which means $x = u_1 + u_2 + u_5 + v_2$.), and $g^{(3)} = w_1 w_3 w_6$ (*i.e.*, $\alpha^{(3)} = (1\ 0 1\ 0\ 0 1)^T$ which means $x = u_1 + u_3 + u_6 + v_2$).

These three solutions, $\alpha^{(1)}, \alpha^{(2)}, \alpha^{(3)}$, are all what we wanted to have in this example.

⑦ Note that we could also obtain $\alpha^{(2)}$ and $\alpha^{(3)}$ directly by using Maple 7 in which only $g_{25},\cdots,g_{30}$ out of g are used to find $g = \overline{f}^{\mathrm{g}}$, but allowing them to change ordering in ⑤.

6. Conclusion

In this paper, dropping minimality assumption, we have applied general IP problems of section 4.2 and special IP problems of section 4.1 to two typical kinds of examples in Petri net problems, by using Groebner basis approach and a symbolic computation system; maple®7[6]. Through even simple and small examples, we could confirm that Groebner basis approach can give all exact solutions which we have also obtained by the other indirect methods[7].

However, we think that there remains a lot of future problems which can be checked, for example, in the Petri net world; how to solve general IP problems with minimality assumption [2], [3], how to select Groebner basis with different monomial orderings, which computer algebra system among available symbolic computation systems should be adopted, and so on.

7. References

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